1. The Geometry of Curves

This quarter we will study curves in $\mathbb{R}^2$ and $\mathbb{R}^3$. A parametric curve in $\mathbb{R}^3$ is a map

$$\begin{aligned} \alpha : \mathbb{R} &\longrightarrow \mathbb{R}^3 \\ t &\longmapsto \alpha(t) := (x(t), y(t), z(t)). \end{aligned}$$

We say that α is differentiable if x, y, z are differentiable. We say that it is C^1 if, in addition, the derivatives are continuous. We say α is C^n if the first n derivatives exist and are continuous. We say that α is C^∞ or smooth if derivatives to any order exist.

The length of a parametric curve from t_1 to t_2 is the integral

$$\int_{t_1}^{t_2} |\alpha'(t)| dt$$

where $|\alpha'(t)| := \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}$ is the length of the *velocity vector* $\alpha'(t) := (x'(t), y'(t), z'(t))$. Cut up the interval into small pieces and take the limit as the lengths of the small intervals go to zero to justify it.

The length of the vector $v := (a_1, b_1, c_1)$ is $|v| := \sqrt{a_1^2 + b_1^2 + c_1^2}$ which is also $\sqrt{v \cdot v}$ where $\cdot$ is the scalar or dot product. We have

$$v \cdot w := a_1 a_2 + b_1 b_2 + c_1 c_2.$$

Also,

$$v \cdot w = |v|\,|w| \cos(\theta)$$

where θ is the angle between v and w. To see this, draw the triangle with sides v, w and $v - w$ and drop a perpendicular from one vertex to the opposing side, then use the Pythagorean theorem and the fact that some ratios are $\cos(\theta)$ and some are $\sin(\theta)$. The above leads to the **Schwarz inequality**:

$$|v \cdot w| \le |v|\,|w|.$$

Theorem 1.1. *A line is the shortest curve between two points.*

Proof. Let v and w be two points of $\mathbb{R}^3$, also thought of as vectors starting from the origin. Parametrize the line joining them as $l(t) := v + t(v - w)$. Then the length of the line is

$$L(l) = \int_0^1 |l'(t)| dt = \int_0^1 |v - w| dt = |v - w|.$$

Now, given any curve $\alpha(t)$ from v to w, the length of α is $L(\alpha) = \int_a^b |\alpha'(t)| dt$. We use Schwarz' inequality for the vector $u := \frac{w-v}{|w-v|}$ which has length 1:

$$|\alpha'(t)| = |\alpha'(t)|\,1 \geq |\alpha'(t)|\,1\,|\cos(\theta)| = |\alpha'(t) \cdot u|.$$

and obtain

$$L(\alpha) \geq \int_a^b |\alpha'(t) \cdot u|\,dt \geq \int_a^b \alpha'(t) \cdot u\,dt = (\alpha(b) - \alpha(a)) \cdot u = (w - v) \cdot u = |w - v|.$$

□

2. Arclength parametrization

We say that a curve is parametrized by arclength s if the parameter s is the length of the curve from a certain point. Let $s(t) := \int_a^t |\alpha'(t)|\,dt$ be the length of the curve from the point $t = a$.

A reparametrization of the curve is a composition $\beta(t) = \alpha(u(t))$ where u is a function of a real variable. We say a curve is *regular* if $\alpha'(t)$ is never 0.

To reparametrize a curve by arclength, we must have

$$s(t) = t = \int_a^t |\beta'(t)|\,dt.$$

Taking derivatives of both sides, we obtain $|\beta'(t)| = 1$ by the fundamental theorem of Calculus. Conversely, if $|\beta'(t)| = 1$, then $s(t) = t$ and the curve is parametrized by arclength.

So to parametrize a curve by arclength means finding a parametrization such that the velocity vector always has length 1. We have

Theorem 2.1. *Every regular curve can be reparametrized by arclength.*

Proof. Since the arclength is defined by

$$s(t) = \int_a^t |\beta'(t)|\,dt,$$

we have, by the fundamental theorem of Calculus, $s'(t) = |\alpha'(t)|$. In particular, $s'(t)$ is always nonnegative. Since the curve is regular, $|\alpha'(t)|$ is never 0, hence $s(t)$ is strictly increasing and therefore one-to-one. So $s(t)$ has an inverse function $t(s)$ which is also strictly increasing. Defining $\beta(s) := \alpha(t(s))$, we have

$$|\beta'(s)| = |\alpha'(t)\frac{dt}{ds}| = \frac{|\alpha'(t)|}{ds/dt} = 1$$

and we have a parametrization by arclength. □

3. Frenet frames

A Frenet frame can be thought of as the path along which you would want three fingers to grab a curve so that it has no way of escaping! It is a coordinate system that moves on the curve. The first vector is the unit tangent vector. So it is most convenient to assume that the curve is parametrized by arclength so that the velocity vector automatically has length 1. The other two vectors should be perpendicular to the tangent vector but how should we choose them? They should be determined by the curve. One important property of the curve is how it turns. So the second vector will be a unit vector that tells us how the tangent vector changes. We define

$$N(t) := \frac{T'(t)}{|T'(t)|},$$

Assuming that the curvature

$$\kappa(s) := |T'(s)|$$

is not zero. The curvature tells us how fast the tangent vector turns. Note that by the product formula, because the length of T is constant, N is normal to T everywhere.

Definition 3.1. *The osculating plane at the point of parameter s is the plane spanned by the vectors $T(s)$ and $N(s)$.*

The third vector is the unique vector that completes the system T, N into a coordinate system. This vector is

$$B := T \times N$$

the cross-product of T and N. Recall that the cross-product of two vectors $v = (a_1, b_1, c_1)$ and $w = (a_2, b_2, c_2)$ can be defined as the "determinant"

$$v \times w = \begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

where i, j, k are the unit vectors $(1,0,0), (0,1,0), (0,0,1)$ respectively. The length of the cross-product is $|T||N|\sin(\theta)$. So B has length 1. The cross-product is bilinear, non associative:

$$A \times (B \times C) = (A \cdot C)B - (A \cdot B)C \neq (A \times B) \times C,$$

anti-commutative. There is a product rule. If both v and w have length 1, their cross-product is the unit vector perpendicular to both of them, oriented with the thumb-screw rule.

The set (T,N,B) is the Frenet frame along β. We say that (T,N,B) is an orthonormal basis of $\mathbb{R}^3$. The variation of T,N and B will tell us how β twists and turns through space. Their variations are determined by their derivatives T',N' and B'.

We already know $T' = \kappa N$ by the definition of N. We need to find N' and B'. Since (T,N,B) is a basis of $\mathbb{R}^3$, we can write

$$N' = a_1 T + b_1 N + c_1 B \quad B' = a_2 T + b_2 N + c_2 B$$

for some unique real numbers $(a_1,b_1,c_1),(a_2,b_2,c_2)$. Because we have an orthonormal basis, we have $a_1 = N' \cdot T$ etc.

First note that $N \cdot N = B \cdot B = 1$ so, applying the product rule, we obtain

$$N' \cdot N = B' \cdot B = 0.$$

Next $N \cdot T = B \cdot T = 0$. Apply the product rule again to obtain

$$N' \cdot T + N \cdot T' = B' \cdot T + B \cdot T' = 0$$

and

$$N' \cdot T = -N \cdot T' = -N \cdot (\kappa N) = -\kappa$$

and

$$B' \cdot T = -B \cdot T' = -B \cdot (\kappa N) = 0.$$

Finally, $N \cdot B = 0$ gives us in the same way

$$N' \cdot B = -B' \cdot N.$$

This is the only unknown quantity and it is, by definition, the torsion of the curve:

$$\tau \coloneqq N' \cdot B = -B' \cdot N.$$

So putting everything together, we have

$$\begin{array}{rclll} T' & = & & \kappa N & \\ N' & = & -\kappa T & & +\tau B \\ B' & = & & -\tau N & \end{array}$$

So the torsion is the component of the derivative of the normal vector N on the vector B. It tells how the curve twists out of its osculating plane or how twisted the curve is! Torsion is the noun associated to the French verb "tordre" which means to twist.

Examples:

(1) a circle:

$$\beta(s) = P + r\left(\cos\left(\frac{s}{r}\right), \sin\left(\frac{s}{r}\right)\right).$$

then $\kappa = \frac{1}{r}$

(2) a helix:

$$\alpha(s) = \left(a\cos\left(\frac{s}{c}\right), a\sin\left(\frac{s}{c}\right), \frac{b}{c}\,s\right) \qquad c = \sqrt{a^2+b^2}.$$

Recall that $\kappa = |T'|$ is always ≥ 0.

Theorem 3.2. *Let β be a unit speed curve. Then*

(1) *$\kappa = 0$ if and only if β is a line.*

(2) *for $\kappa > 0$, $\tau = 0$ if and only if β is a plane curve.*

Proof. Read. □

Theorem 3.3. *A curve β is part of a circle if and only if it is a plane curve ($\tau = 0$) and $\kappa > 0$ is constant.*

Proof. Suppose β is part of a circle. Then $\tau = 0$ because β is a plane curve. Let P be the center of the circle. Parametrize the circle as in the example above to get $\kappa = \frac{1}{r}$ is a positive constant.

Conversely, suppose $\tau = 0$, i.e., β is a plane curve, and κ is a positive constant. To show that the curve is part of a circle, we need to find the center of the circle. For any circle, the curvature is the inverse of the radius. So $\beta + \frac{1}{\kappa}N = P$ would be the center of the circle. This means we need to show $\beta + \frac{1}{\kappa}N$ is constant. Differentiate to obtain

$$T + \frac{1}{\kappa}N' = T + \frac{1}{\kappa}(-\kappa T + \tau B) = 0.$$

which is what we needed. □

Definition 3.4. *For a general plane curve, the curve defined by*

$$\gamma(s) = \beta(s) + \frac{1}{\kappa(s)}N(s)$$

is the evolute of β: the locus of centers of its osculating circles.

4. Non unit speed curves

As we saw, it is not always easy to reparametrize a curve by arclength so we need to modify our Frenet formulas so that we can also compute with arbitrary parametrizations.

For this we have to recall

$$\beta(s) = \alpha(t).$$

The Frenet frame is defined in terms of a unit speed parametrization. So we theoretically switch to unit speed and then use the Chain rule to compute with an arbitrary parametrization. $T(s) = T(t)$ is always the unit tangent vector. So

$$T(t) = \frac{\alpha'(t)}{|\alpha'(t)|} = \beta'(s).$$

Recall that the curvature is

$$\kappa(s) := \left|\frac{dT}{ds}\right|.$$

If we don't have an explicit arclength parametrization, we can take derivatives with respect to t. Recall the formulas

$$\begin{aligned}
\tfrac{dT}{ds} &= & &\kappa N & \\
\tfrac{dN}{ds} &= &-\kappa T & & +\tau B\\
\tfrac{dB}{ds} &= & &-\tau N. &
\end{aligned}$$

Using the chain rule, we obtain

$$\begin{aligned}
\tfrac{dT}{dt} = \tfrac{dT}{ds}\tfrac{ds}{dt} &= & &\kappa \tfrac{ds}{dt} N & \\
\tfrac{dN}{dt} = \tfrac{dN}{ds}\tfrac{ds}{dt} &= &-\kappa \tfrac{ds}{dt} T & & +\tau \tfrac{ds}{dt} B\\
\tfrac{dB}{dt} = \tfrac{dB}{ds}\tfrac{ds}{dt} &= & &-\tau \tfrac{ds}{dt} N. &
\end{aligned}$$

First note that

$$\kappa = \left|\frac{dT}{ds}\right| = \frac{1}{\nu(t)}\left|\frac{dT}{dt}\right|.$$

Recall that $\frac{ds}{dt} = |\alpha'(t)| = \nu(t)$ is the speed of the particle. We have

$$\alpha'(t) = |\alpha'(t)|T(t) = \frac{ds}{dt}T(t) = \nu(t)T(t),$$

so

$$\alpha''(t) = \nu'(t)T(t) + \nu(t)T'(t) = \nu'(t)T(t) + \kappa(t)\nu(t)^2 N,$$

and

$$\alpha'(t) \times \alpha''(t) = \kappa(t)\nu(t)^3 T(t) \times N(t) = \kappa\nu^3 B.$$

This gives us both κ and B:

$$\kappa = \frac{|\alpha' \times \alpha''|}{\nu^3}, \qquad B = \frac{\alpha' \times \alpha''}{|\alpha' \times \alpha''|}$$

and we also obtain N as

$$N = B \times T$$

and τ as

$$\tau = \frac{N' \cdot B}{\nu}.$$

To make τ more explicit in terms of the derivatives of α, compute

$$\alpha''' = (\nu' T + \kappa \nu^2 N)' = \nu'' T + \nu' T' + \kappa' \nu^2 N + 2\kappa \nu \nu' N + \kappa \nu^2 N'.$$

Then (note $T' = \kappa \nu N$)

$$(\alpha' \times \alpha'') \cdot \alpha''' = \big(\nu T \times (\nu' T + \kappa \nu^2 N)\big) \cdot \big(\nu'' T + \nu' T' + \kappa' \nu^2 N + 2\kappa \nu \nu' N + \kappa \nu^2 N'\big)$$

$$= \kappa^2 \nu^5 B \cdot N' = \tau \kappa^2 \nu^6.$$

So

$$\tau = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{\kappa^2 \nu^6} = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{|\alpha' \times \alpha''|^2}$$

since $\kappa \nu^3 = |\alpha' \times \alpha''|$.

4.1. Recall (Exercise 1.2.7) that the *involute* of a plane curve $\alpha(t)$ is given by

$$\mathcal{I}(t) = \alpha(t) - s(t) \frac{\alpha'(t)}{|\alpha'(t)|} = \alpha(t) - s(t) T(t)$$

and that the *evolute* of a plane curve is given by

$$\mathcal{E}(t) = \alpha(t) + \frac{N(t)}{\kappa(t)}.$$

These two operations are inverses to each other (read the proof in the book).

4.2. Example. $\alpha(t) = (t, t^2, 0)$ (question: do you recognize this curve?). Compute Frenet frames, curvature and torsion. Compute the evolute and the involute of the evolute.

5. The Fundamental theorem (1.5.17)

Theorem 5.1. *Given smooth functions $\kappa(s) > 0$ and $\tau(s)$ on an interval $I \subset \mathbb{R}$, there exists a regular arclength parametrized curve $\alpha : I \to \mathbb{R}$ such that κ is its curvature and τ its torsion. Such a curve is unique up to a rigid motion (i.e. a combination of a rotation and a translation) of $\mathbb{R}^3$.*

Proof. (sketch of proof of existence, proof of uniqueness) The proof uses existence theorems for differential equations: one solves the differential equations given by the Frenet formulas, i.e., finds the vector valued functions T, N, B satisfying those equations. Then one defines the curve as

$$\alpha(s) := \int_0^s T(\sigma) d\sigma.$$

Uniqueness is proved separately. Given two curves α_1, α_2 with the same curvature and torsion, first choose a point s_0 and translate one curve so as to obtain $\alpha_1(s_0) = \alpha_2(s_0)$. Use a rotation to make the Frenet frames agree at s_0. Now show that the two curves agree everywhere by computing

$$\begin{gathered}\frac{d}{ds}\left(|T_1 - T_2|^2 + |N_1 - N_2|^2 + |B_1 - B_2|^2\right) \\ = 2\left((T_1 - T_2)(T_1' - T_2') + (N_1 - N_2)(N_1' - N_2') + (B_1 - B_2)(B_1' - B_2')\right) \\ = 2\left(\kappa(T_1 - T_2)(N_1 - N_2) - \kappa(N_1 - N_2)(T_1 - T_2) + \tau(N_1 - N_2)(B_1 - B_2) - \tau(B_1 - B_2)(N_1 - N_2)\right) \\ = 0\end{gathered}$$

So this is constant and, since it is zero at s_0, it is zero everywhere. So the frames agree everywhere. Next

$$(\alpha_1 - \alpha_2)' = T_1 - T_2 = 0$$

so $\alpha_1 - \alpha_2$ is also constant. It is zero at s_0, so it is zero everywhere. □

6. Green's theorem and the isoperimetric inequality

A smooth curve $\alpha : [a, b] \to \mathbb{R}$ is closed if $\alpha^{(n)}(a) = \alpha^{(n)}(b)$ for all $n \geq 0$. Closed curves often represents periodic orbits of physical systems. A closed curve is simple if it does not cross itself, in other words, α is one-to-one on $[a, b)$.

Theorem 6.1. *(The isoperimetric inequality) Let C be a simple closed curve of length L, bounding a region of area A. Then*

$$L^2 \geq 4\pi A$$

with equality exactly when C is a circle.

For the proof we will need the formula

$$A = \int_a^b x(t)y'(t)dt = -\int_b^a y(t)x'(t)dt.$$

If the curve can be divided into the union of the graphs of two functions f and g, then

$$A = \int_{x_1}^{x_2} g(x)dx - \int_{x_1}^{x_2} f(x)dx = -\int_{t_2}^{t_1} y(t)x'(t)dt - \int_{t_1}^{b} y(t)x'(t)dt - \int_{a}^{t_2} y(t)x'(t)dt = \int_a^b y(t)x'(t)dt.$$

Integration by parts yields the second formula:

$$\int_a^b y(t)x'(t)dt = [-y(t)x(t)]_a^b + \int_a^b x(t)y'(t)dt.$$

Or one can use

Theorem 6.2. *(Green's theorem) Let P and Q be smooth functions on a simply connected region (i.e., a region without holes) R of the plane with boundary a simple closed curve C. Then*

$$\iint_R \left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) dxdy = \int_C (Pdx - Qdy).$$

Proof. (of the isoperimetric inequality) Take parallel lines L, L' on the two sides of C and move them towards C until they touch, at which point they will be tangent to C. Choose them in such a way that they are not parallel to any straight segments of C. Also choose a circle S of radius $r =$ half of the distance between L and L' and tangent to both. Choose the origin at the center of the circle S with the y-axis parallel to L and L'. Parametrize C by arclength: $\alpha(s) = (x(s), y(s))$ with $s \in [0, L]$ and points of tangency to L, L' at $s = 0, s = s_0$.

The points of tangency divide C and S into upper and lower arcs and we can map each point of C to a point of S with the same x coordinate on the corresponding arc. This gives a parametrization $\beta(s) = (x(s), \overline{y}(s) = \pm\sqrt{r^2 - x(s)^2})$ of the circle S where the coordinate $\overline{y}$ is negative on the lower arc $s \in [0, s_0]$ and positive on the upper arc $s \in [s_0, L]$. Note that this is not arclength parametrization on the circle and the point might go back and forth on the circle.

The areas of C and S are

$$A = \int_0^L x(s)y'(s)ds, \quad \pi r^2 = -\int_0^L \overline{y}(s)x'(s)ds.$$

So

$$A + \pi r^2 = \int_0^L (xy' - \overline{y}x')ds \leq \int_0^L |xy' - \overline{y}x'|\, ds = \int_0^L \sqrt{(xy' - \overline{y}x')^2} ds.$$

Now note

$$(xy' - \overline{y}x')^2 = (x^2 + \overline{y}^2)(x'^2 + y'^2)^2 - (xx' + \overline{y}y')^2 \leq (x^2 + \overline{y}^2)(x'^2 + y'^2)^2 = x^2 + \overline{y}^2$$

because α has unit speed. So

$$A + \pi r^2 \le \int_0^L \sqrt{(x^2 + \overline{y}^2)}ds = \int_0^L r ds = Lr.$$

Now the arithmetic-geometric mean inequality, obtained from $(a-b)^2 \ge 0$ gives

$$\sqrt{A\pi r^2} \le \frac{1}{2}\left(A + \pi r^2\right) \le \frac{1}{2}Lr$$

and

$$4\pi A \le L^2.$$

To have equality, all the inequalities we used must be equalities. For the arithmetic-geometric mean, this means $A = \pi r^2$ and $L = 2\pi r$.

We also have $xx' + \overline{y}y' = 0$ and $xy' - \overline{y}x' = r$. Hence $xx'^2 = -\overline{y}y'x'$ and $xy'^2 - \overline{y}y'x' = ry'$. So $x = x(x'^2 + y'^2) = ry'$. Now replace y' with x/r in the first equation to get $\overline{y} = -rx'$. So, on the upper circle, $\sqrt{r^2 - x^2} = -rx'$ and

$$\int \frac{dx}{\sqrt{r^2 - x^2}} = -\int \frac{ds}{r}.$$

If θ is the angle on the circle, we have $x = r\sin\theta$ and

$$\theta = -\frac{s}{r} + d, \quad x = r\sin\left(-\frac{s}{r} + d\right).$$

Next

$$y = \int \frac{x}{r} ds = r\cos\left(\frac{s}{r} + d\right).$$

Similarly on the lower circle. $\square$

7. Surfaces

This quarter, surfaces for us will be unions of images of C^∞ (partials with respect to the two variables to any order exist) maps

$$\varphi : D \longrightarrow \mathbb{R}^3$$

where D is an open (meaning D does not contain its boundary) domain in $\mathbb{R}^2$, usually an open rectangle $]a,b[\times]c,d[$ or an open disc $|(x,y) - P| < r$. If u and v are the coordinates on D, we write $\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$ and define

$$\varphi_u := \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right), \quad \varphi_v := \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right).$$

A map φ is called *regular* if the derivatives φ_u, φ_v are linearly independent everywhere on D, i.e., the cross-product $\varphi_u \times \varphi_v$ is everywhere nonzero. The vector $\varphi_u \times \varphi_v$ is a normal to the surface and we can obtain a unit normal by dividing by its length.

We can usually make D smaller so that the map φ is one-to-one and regular. Such a map φ is then called a *coordinate chart or patch* for the surface. For such a map φ there exists an inverse (defined on the image of φ). If we have two coordinate charts φ and ψ we can form the composition

$$\psi^{-1} \circ \varphi : D \longrightarrow \mathbb{R}^2.$$

A surface is called *smooth* if for all coordinate charts φ, ψ, the composition $\psi^{-1} \circ \varphi$ is smooth. The composition $\psi^{-1} \circ \varphi$ is a *change of coordinates.*

A set of coordinate charts covering S is usually called an *atlas.*

7.1. Graphs (or Monge patches). $\varphi(u,v) = (u, v, f(u,v))$ for a function of two variables f. Always a coordinate patch if f is smooth.

Examples: the paraboloid: $f(u,v) = u^2 + v^2$; the saddle: $z = xy$.

Paraboloid: $z =$ constant, we get circles, x or $y =$ constant, we get parabolas.

7.2. The implicit function theorem. When is an implicit equation $f(x,y,z) = c$ a surface? If we could solve for z, we would be able to write $z = g(x,y)$. Then on the surface,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0$$

and

$$dz = \frac{-\left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right)}{\frac{\partial f}{\partial z}}.$$

Theorem 7.1. *(The implicit function theorem) Near any point (a,b,c) satisfying $f(x,y,z) = c$ and $\frac{\partial f}{\partial z} \neq 0$, z can be written as a smooth function of x and y whose partials are*

$$\frac{\partial z}{\partial x} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}, \quad \frac{\partial z}{\partial y} = \frac{-\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}.$$

c is called a regular value of f, if no point satisfies $f(x,y,z) = c$, $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$.

7.3. Spheres. $x^2 + y^2 + z^2 = 1$. Upper, lower, left, right hemispheres... $z = \sqrt{x^2 + y^2}$ etc.

Spherical coordinates: $\omega(\theta, \phi) = R(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$.

Geographical coordinates: $u = \theta$, $v = \pi/2 - \phi$.

Compute the derivatives and the unit normal. What points on the sphere do we miss? Are these coordinate patches? How should we define the domains D?

7.4. Surfaces of revolution. Regular curve $\alpha(u) = (g(u), h(u), 0)$. Rotate about the x-axis:

$$\varphi(u, v) = (g(u), h(u)\cos v, h(u)\sin v).$$

How should we define D? Compute unit normal in homework.

7.5. Ruled surfaces. A surface is ruled if it has a parametrization

$$w(u, v) = \alpha(u) + v\beta(u)$$

where α and β are two curves. This may not be a coordinate patch, we might have to remove some points to get coordinate patches. Each time we fix u, we get a line in S.

Examples:

Cones: $w = P + v\beta$ where P is fixed (all the lines go through one point).

Saddles: $z = xy$ (doubly ruled).

Hyperboloid of one sheet: $x^2 + y^2 - z^2 = 1$ (doubly ruled, see exercises in the book).

8. The geometry of surfaces

Curves have tangent lines, surfaces have tangent planes. There are 3 equivalent ways of looking at the tangent spaces of a surface:

(1) The tangent plane to S at the point (a, b, c) is the union of all the tangent lines to curves in S through (a, b, c).

(2) The tangent plane is the plane through (a, b, c) spanned by the vectors φ_u, φ_v in a coordinate chart φ near (a, b, c).

(3) The tangent plane is the plane through (a, b, c) with normal vector $\varphi_u \times \varphi_v$.

The first one is not so easy to compute with useful for geometric arguments and it is nice because it shows that the tangent plane is independent of the choice of chart.

A curve defined on an interval $I = [a,b]$ lies on a surface S if the map $\alpha : I \to \mathbb{R}^3$ factors through S. In other words, $\alpha : I \to S \subset \mathbb{R}^3$. The datum of a curve in S is equivalent to the data of two smooth maps $u(t), v(t) : I \to \mathbb{R}$ such that $\alpha(t) = \varphi(u(t), v(t))$. (See Lemma 2.1.3)

To see that the first and second are equivalent, consider a curve $\alpha : I \to \mathbb{R}^3$ in S through (a,b,c). Then $\alpha(t) = \varphi(u(t), v(t))$ and we have

$$\alpha'(t) = \varphi_u u' + \varphi_v v'$$

Hence the vector α' belongs to the span of φ_u and φ_v. Conversely, suppose given a linear combination $\lambda\varphi_u + \mu\varphi_v$. Define a curve $\alpha(t) = (u_0 + t\lambda, v_0 + t\mu)$ and take t small enough that $(u_0 + t\lambda, v_0 + t\mu) \in D$. Then the tangent vector to this curve is $\lambda\varphi_u + \mu\varphi_v$.

The second and third one are equivalent because the span of φ_u, φ_v is perpendicular to the cross product of the two vectors.

Recall that the equation of a plane can be written using the dot product:

$$N \cdot (Q - P) = 0$$

where N is a normal vector to the plane, P is a fixed point on the plane and Q is a variable point on the plane.

We define the unit normal vector

$$U := \frac{\varphi_u \times \varphi_v}{|\varphi_u \times \varphi_v|}.$$

Definition 8.1. *A surface is orientable if it has a consistent normal vector defined everywhere on it. Single coordinate patches are always orientable. Problems might occur when we have to use more than one patch: we need to be able to glue the normals from one patch to the other.*

Essentially, the surface is orientable if, when we are moving on any loop (simple closed curve), then the normal at the starting point is equal to the normal at the ending point.

Example:: The Möbius strip is not orientable. First patch

$$\varphi(u_1, v_1) = \left(\left(2 - v_1 \sin\left(\frac{u_1}{2}\right)\right)\sin u_1, \left(2 - v_1 \sin\left(\frac{u_1}{2}\right)\right)\cos u_1, v_1 \cos\left(\frac{u_1}{2}\right)\right)$$

where $u_1 \in]0, 2\pi[, v_1 \in]-1, 1[$. Second patch

$$\psi(u_2, v_2) = \left(\left(2 - v_2 \sin\left(\frac{\pi}{4} + \frac{u_2}{2}\right)\right)\sin u_2, -\left(2 - v_2 \sin\left(\frac{\pi}{4} + \frac{u_2}{2}\right)\right)\cos u_2, v_2 \cos\left(\frac{\pi}{4} + \frac{u_2}{2}\right)\right)$$

where $u_2 \in]0, 2\pi[, v_2 \in]-1, 1[$.

From now on we assume that our surfaces are orientable.

Definition 8.2. *The shape operator S for a surface M is defined as*

$$S_p(V) = -\nabla_V U$$

at a point $p \in M$ where $\nabla_V U$ is the directional derivative of U in the direction of V. In other words, if $U = (f, g, h)$ and $V = (a, b, c)$, then

$$\nabla_V U = (\nabla_V f, \nabla_V g, \nabla_V h) = \left(a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} + c\frac{\partial f}{\partial z}, a\frac{\partial g}{\partial x} + b\frac{\partial g}{\partial y} + c\frac{\partial g}{\partial z}, a\frac{\partial h}{\partial x} + b\frac{\partial h}{\partial y} + c\frac{\partial h}{\partial z}\right).$$

Theorem 8.3. *(Lemma 2.2.10) S_p is a linear transformation from T_pM to itself.*

Proof. We have to see that S_p sends T_pM into T_pM. For this we verify that

$$S_p(V) \cdot U = 0$$

for all $V \in T_pM$. We have the usual Leibniz rule

$$0 = \nabla_V(1) = \nabla_V(U \cdot U) = 2U \cdot \nabla_V(U).$$

Verify linearity. □

Note that if M is contained in a plane, then U is constant and all its derivatives are 0. We say that S is path connected, if for any two points $p, q \in S$, there exists a curve $\alpha : [0, 1] \to S$ such that $\alpha(0) = p$ and $\alpha(1) = q$. Conversely

Theorem 8.4. *Assume M is path connected. If $S_p = 0$ at every point of M, then M is contained in a plane.*

Proof. Fix a point $p \in M$ with unit normal $U(p)$. We will show that any other point $q \in M$ belongs to the plane through p with normal $U(p)$. This means

$$(q - p) \cdot U(p) = 0.$$

Choose a curve α in M with $\alpha(0) = p$ and $\alpha(1) = q$ and define

$$f(t) = (q - \alpha(t)) \cdot U(\alpha(t)).$$

Compute

$$f'(t) = -\alpha'(t) \cdot U(\alpha(t)) + (q - \alpha(t)) \cdot U'(\alpha(t)) = -\alpha'(t) \cdot U(\alpha(t)) + (q - \alpha(t)) \cdot \nabla_{\alpha'(t)} U(\alpha(t)) = 0$$

because $\alpha'(t) \in T_{\alpha(t)}M$ is perpendicular to $U(\alpha(t))$ and $\nabla_{\alpha'(t)}U(\alpha(t)) = -S_{\alpha(t)}(\alpha'(t)) = 0$. Therefore f is constant and

$$0 = f(1) = f(0) = (q - p) \cdot U(p).$$

□

Examples:

The sphere of radius R: $\varphi(u, v) = R(\cos u \cos v, \sin u \cos v, \sin v)$. $S_p(V) = -\frac{V}{R}$ for all V.

9. The linear algebra of surfaces

The computation of the shape operator doesn't always give us geometric information about the surface, i.e., its shape (see the example of the saddle, homework 2.2.16). However, there are other things we can do with the shape operator that do give us information about the shape of the surface. To study the shape operator better, we first need to review a few notions from linear algebra.

Matrix of a linear transformation on a basis.

Eigenvalues and eigenvectors. Matrix on a basis of eigenvectors and diagonalization.

Determinant and Trace in terms of eigenvalues.

Definition 9.1. *We say that a linear operator T is symmetric if, for any vectors v and w, $T(v) \cdot w = v \cdot T(w)$.*

Definition 9.2. *(reminder) A basis is orthonormal if it consists of orthogonal (perpendicular) unit vectors.*

Exercise 2.3.4: With respect to an orthonormal basis, the matrix of a symmetric operator is symmetric.

Theorem 9.3. *(2.3.5) We have*

$$S(\varphi_u) \cdot \varphi_u = \varphi_{uu} \cdot U, \quad S(\varphi_u) \cdot \varphi_v = \varphi_{uv} \cdot U = S(\varphi_v) \cdot \varphi_u, \quad S(\varphi_v) \cdot \varphi_v = \varphi_{vv} \cdot U.$$

In particular, the shape operator is symmetric.

Proof. The symmetry follows from the equations by linearity (write it out for them). Note that $U \cdot \varphi_u = U \cdot \varphi_v = 0$. Take derivatives with respect to u and v:

$$0 = \frac{\partial}{\partial u}(U \cdot \varphi_u) = U_u \cdot \varphi_u + U \cdot \varphi_{uu}, \quad 0 = \frac{\partial}{\partial v}(U \cdot \varphi_u) = U_v \cdot \varphi_u + U \cdot \varphi_{uv}$$

$$0 = \frac{\partial}{\partial u}(U \cdot \varphi_v) = U_u \cdot \varphi_v + U \cdot \varphi_{uv}, \quad 0 = \frac{\partial}{\partial v}(U \cdot \varphi_v) = U_v \cdot \varphi_v + U \cdot \varphi_{vv}$$

□

The shape operator S_p contains information about the acceleration/curvature of curves in M through p.

Lemma 9.4. *(2.4.1) For any curve $\alpha(t)$ in M,*

$$\alpha'' \cdot U = S(\alpha') \cdot \alpha'.$$

Proof. (third proof in the book) Take the derivative of $U \cdot \alpha' = 0$ with respect to the parameter t:

$$0 = \frac{d}{dt}(U \cdot \alpha') = U' \cdot \alpha' + U \cdot \alpha''.$$

By the definition of S, $U' = -S(\alpha')$. □

The scalar product $\alpha'' \cdot U$ is the part of the acceleration due to the bending of M. The above formula shows that it only depends on the tangent vector to the curve. If the curve has unit speed, it only depends on the tangent direction to the curve.
If α has unit speed and α' is an eigenvector of S_p, then

$$S_p(\alpha') = \lambda\alpha'$$

and

$$\lambda = \lambda\alpha' \cdot \alpha' = S_p(\alpha') \cdot \alpha' = \alpha'' \cdot U.$$

Definition 9.5. *Given a unit vector $\mathbf{u}$ in the tangent space T_pM, the (normal) curvature of M in the direction of $\mathbf{u}$ is*

$$k(\mathbf{u}) := S_p(\mathbf{u}) \cdot \mathbf{u}.$$

This is the curvature of the curve $P \cap M$ where P is the plane through p and parallel to $U(p)$ and $\mathbf{u}$ (see below and 2.4.3 in the book).
The (real) eigenvalues k_1, k_2 of S_p are the principal curvatures of M at p. The corresponding unit (orthogonal) eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$ are the principal directions.
The determinant $K := k_1k_2$ is the Gaussian curvature at p.
Half the trace $H := \frac{1}{2}(k_1 + k_2)$ is the mean curvature at p.
A point where $k_1 = k_2$ is umbilic: $S_p = k_1 id$ and every direction is principal.

Note that S, k_1, k_2, H change sign under a change of orientation but not K.

The normal curvature is the normal component of acceleration: by the Frenet formulas, for a unit speed curve

$$k(\alpha') = S_p(\alpha') \cdot \alpha' = \alpha'' \cdot U = \kappa N \cdot U = \kappa \cos\theta$$

where θ is the angle between the surface normal and the curve normal (draw a picture).

Proposition 9.6. *(2.4.3) Let* $\mathbf{u}$ *be a unit vector and* P *the plane through* p *and parallel to* $U(p)$ *and* $\mathbf{u}$*. Let* σ *be the unit speed curve formed by* $P \cap M$ *with* $\sigma(0) = p$*. Then*

$$k(\mathbf{u}) = \pm\kappa_\sigma(0).$$

Proof. In the formula above, $\theta = 0$ or π. □

Definition 9.7. *Curves as above are called* normal sections *of* M*.*

Example: the sphere: we saw

$$S_p = \begin{pmatrix} -\frac{1}{r} & 0 \\ 0 & -\frac{1}{r} \end{pmatrix} \quad \text{or} \quad S_p = \begin{pmatrix} \frac{1}{r} & 0 \\ 0 & \frac{1}{r} \end{pmatrix}$$

depending on the orientation (geographic or spherical coordinates). So every point is umbilic. Normal sections of the sphere are great circles.

The sign of the normal curvature tells us whether the surface curves away or towards the normal (the normal to the curve $P \cap M$ points either in the same direction as U or in the opposite direction). If the normal curvature is zero, then the bending is small near the point (in the direction of the unit vector $\mathbf{u}$).

The principal directions are intrinsic to the surface at non-umbilic points, i.e., do not depend on choices of charts unlike the vectors φ_u, φ_v.

Lemma 9.8. *(Euler's formula, 2.4.11) For any unit tangent vector* $\mathbf{u} = \cos\theta\, \mathbf{u}_1 + \sin\theta\, \mathbf{u}_2$*, we have*

$$k(\mathbf{u}) = k_1 \cos^2\theta + k_2 \sin^2\theta$$

Proof. We have

$$k(\mathbf{u}) = S_p(\mathbf{u}) \cdot \mathbf{u} = S_p\left(\cos\theta\, \mathbf{u}_1 + \sin\theta\, \mathbf{u}_2\right) \cdot \left(\cos\theta\, \mathbf{u}_1 + \sin\theta\, \mathbf{u}_2\right)$$

$$= (k_1 \cos\theta\, \mathbf{u}_1 + k_2 \sin\theta\, \mathbf{u}_2) \cdot (\cos\theta\, \mathbf{u}_1 + \sin\theta\, \mathbf{u}_2) = k_1 \cos^2\theta + k_2 \sin^2\theta$$

□

Assuming $k_1 \geq k_2$, we have

$$k(\mathbf{u}) = k_1 \cos^2\theta + k_2 \sin^2\theta = k_1 + (k_2 - k_1)\sin^2\theta \quad \leq k_1$$

and

$$k(\mathbf{u}) = k_1 \cos^2\theta + k_2 \sin^2\theta = k_2 + (k_1 - k_2)\cos^2\theta \quad \geq k_2.$$

So the principal curvatures k_1 and k_2 are the maximum and minimum values of the normal curvatures.

Examples: The cylinder $x^2 + y^2 = 1$ at the point $p = (1,0,0)$. Then $U(p) = (1,0,0)$. Tangent vectors are orthogonal to $U(p)$ so have the form $\mathbf{u} = (0, u_1, u_2)$ where $u_1 = \cos\theta, u_2 = \sin\theta$. A normal for the plane determined by $\mathbf{u}$ (through p and containing $U(p)$) is $(0, -u_2, u_1)$. So the equation of the plane is $-u_2 y + u_1 z = 0$. The intersection of the plane with the cylinder is the set of points of the form $(\sqrt{1-y^2}, y, \frac{u_2}{u_1} y)$. This is an ellipse. Use y as parameter.

A parametrization of the cylinder is $(r\cos u, r\sin u, v)$. The shape operator has eigenvalues $-1/r, 0$.

The torus can be parametrized by $((R+r\cos u)\cos v, (R+r\cos u)\sin v, r\sin u)$. The shape operator has eigenvalues $\frac{-1}{r}$ and $\frac{-\cos u}{R+r\cos u}$.

10. Curvatures

Recall that the sign of K is independent of the choice of orientation, so it is an intrinsic invariant of the surface. Sign of $K = k_1 k_2$: $K > 0$ means all normal sections bend the same way, $K < 0$ means some normal sections bend one way, some bend the other way. Such points are called saddle points. $K = 0$ means one of the principal curvatures is 0, hence there is a normal section with curvature 0, all other sections bend the same way.

Examples: For the torus above, $K > 0$ when $\cos u > 0$, $K < 0$ when $\cos u < 0$ and $K = 0$ when $\cos u = 0$.

For the cylinder $K = 0$ everywhere, $H = \frac{1}{2r}$.

Definition 10.1. (1) *An asymptotic direction at p is one with 0 normal curvature (only exists if $K \leq 0$.*

(2) *An asymptotic curve is a curve for which α' is asymptotic everywhere.*

(3) *A principal curve or line of curvature is a curve where T is $\mathbf{u}_1$ or $\mathbf{u}_2$ everywhere.*

(4) *M is called flat if $K = 0$ everywhere*

(5) *M is called minimal if H is zero everywhere.*

Examples: (1) Cylinders are flat, lines and circles are principal on a cylinder, lines are asymptotic.

(2) Spheres: Curve: horizontal circle at angle θ. Radius = $r\cos\theta$. Parametrization of the circle:

$$\alpha(t) = (r\cos\theta\cos\phi, r\cos\theta\sin\phi, r\sin\theta).$$

By Exercise 2.4.4 (homework) these are lines of curvature or principal curves. This also tells us what the other principal direction is, given that they are orthogonal to each other.

Theorem 10.2. *(3.5.2) If every point of a* connected *surface (i.e., M has only one sheet or piece, or M is path connected) is umbilic, then M is contained in a plane or sphere.*

Proof. We know $S_p = kId$ at every point ($k = k_1 = k_2$) but k might vary from point to point. The first task is to show that k is constant. Every direction is an eigenvector, in particular, φ_u and φ_v for any chart φ. So

$$S_p(\varphi_u) = -U_u = k\varphi_u, \quad S_p(\varphi_v) = -U_v = k\varphi_v.$$

So

$$U_{uv} = U_{vu} = -k\varphi_{uv} - k_v\varphi_u = -k\varphi_{uv} - k_u\varphi_v.$$

So $k_v\varphi_u = k_u\varphi_v$. Since φ_u and φ_v are linearly independent, we obtain $k_u = k_v = 0$ and k is constant.

Case 1: If $k = 0$, then $U_u = U_v = 0$ and U is constant. Now

$$(U\cdot\varphi)_u = U_u\cdot\varphi + U\cdot\varphi_u = 0 = (U\cdot\varphi)_v$$

So $U\cdot\varphi$ is constant which means that the patch is contained in a plane perpendicular to U.

Case 2: If $k \neq 0$, then φ "should" be contained in a sphere of radius $\left|\frac{1}{k}\right|$ and center $\varphi + \frac{1}{k}U$ (visualize the sign: if U points out, then $k = \frac{1}{r} > 0$, if U points in, then $k = -\frac{1}{r} < 0$). So, as in the case of evolutes of plane curves, we take the derivatives of $\varphi + \frac{1}{k}U$ and show that they are 0 to show that $\varphi + \frac{1}{k}U$ is constant.

$$\frac{\partial}{\partial u}\left(\varphi + \frac{1}{k}U\right) = \varphi_u + \frac{1}{k}U_u = 0 = \frac{\partial}{\partial v}\left(\varphi + \frac{1}{k}U\right).$$

So $p := \varphi + \frac{1}{k}U$ is constant and

$$|\varphi - p| = \frac{1}{k}$$

which shows that the patch lies on the sphere of center p and radius $\frac{1}{k}$.

What we did above shows that any given patch lies on a plane or sphere. Because the surface is connected, any two patches intersect so the values of k and U coincide. It follows that if one patch is contained in a plane, then the other is contained in the same plane and if one patch is contained in a sphere, then the other is contained in the same sphere. $\square$

11. Computing curvatures

We are going to learn more efficient ways of computing K and H, k_1, k_2 and the matrix of S_p.

As usual, use the basis φ_u, φ_v for T_pM. Then, if

$$S_p(\varphi_u) = a\varphi_u + b\varphi_v, \quad S_p(\varphi_v) = c\varphi_u + d\varphi_v,$$

the matrix of S_p is

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

and $K = ad - bc, H = \frac{1}{2}(a+d)$. Compute

$$S_p(\varphi_u) \times S_p(\varphi_v) = (ad - bc)\varphi_u \times \varphi_v = K\varphi_u \times \varphi_v.$$

Dot with $\varphi_u \times \varphi_v$ and use Lagrange's identity (homework Exercise 3.1.5):

$$(v \times w) \cdot (a \times b) = (v \cdot a)(w \cdot b) - (v \cdot b)(w \cdot a)$$

to get

$$\begin{aligned} (S_p(\varphi_u) \times S_p(\varphi_v)) \cdot (\varphi_u \times \varphi_v) &= (S_p(\varphi_u) \cdot \varphi_u)(S_p(\varphi_v) \cdot \varphi_v) - (S_p(\varphi_u) \cdot \varphi_v)(S_p(\varphi_v) \cdot \varphi_u) \\ &= K(\varphi_u \times \varphi_v) \cdot (\varphi_u \times \varphi_v) = K((\varphi_u \cdot \varphi_u)(\varphi_v \cdot \varphi_v) - (\varphi_u \cdot \varphi_v)(\varphi_v \cdot \varphi_u)). \end{aligned}$$

As in the book, page 88, we introduce the notation

$$E := \varphi_u \cdot \varphi_u, \quad F := \varphi_u \cdot \varphi_v, \quad G := \varphi_v \cdot \varphi_v$$

and

$$l := S_p(\varphi_u) \cdot \varphi_u = U \cdot \varphi_{uu}, \quad m := S_p(\varphi_u) \cdot \varphi_v = U \cdot \varphi_{uv} = S_p(\varphi_v) \cdot \varphi_u, \quad n := S_p(\varphi_v) \cdot \varphi_v = U \cdot \varphi_{vv}.$$

So

$$K = \frac{ln - m^2}{EG - F^2}.$$

Similarly

$$(S_p(\varphi_u) \times \varphi_v) + \varphi_u \times (S_p(\varphi_v)) = (a+d)\varphi_u \times \varphi_v = 2H\varphi_u \times \varphi_v.$$

Dot with $\varphi_u \times \varphi_v$ and use Lagrange to get

$$H = \frac{Gl + En - 2Fm}{2(EG - F^2)}.$$

Now k_1, k_2 can be found from the formulas in Exercise 3.1.7 (homework):

$$k_i = H \pm \sqrt{H^2 - K}.$$

If necessary, we can compute l, m, n in terms of E, F, G and the matrix of S_p:

$$l = S(\varphi_u) \cdot \varphi_u = aE + bF, \quad m = aF + bG = cE + dF, \quad n = cF + dG.$$

So

$$\begin{pmatrix} l & m \\ m & n \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

or get the matrix of S_p from E, F, G, l, m, n:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} l & m \\ m & n \end{pmatrix}.$$

These give us the formulas (book page 89)

$$a = -\frac{Fm - Gl}{EG - F^2}, \quad b = \frac{-Fl + Em}{EG - F^2}, \quad d = \frac{En - Fm}{EG - F^2}, \quad c = -\frac{-mG + Fn}{EG - F^2}.$$

The matrix

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

is the *first fundamental form* or the *metric* of the surface.

Remark: The curves u = constant, v = constant (the coordinate curves) are principal curves (lines of curvature) if and only if $F = m = 0$ (provided M has only isolated umbilic points, see Exercise 3.2.7 in homework). In such a case the principal curvatures are $a = \frac{l}{E}$ and $d = \frac{n}{G}$.

Examples:

(1) The plane: $\varphi(u,v) = (u,v,0), \varphi_u = (1,0,0), \varphi_v = (0,1,0), U = (0,0,1),\ E = G = 1, F = 0, l = m = n = 0, K = H = 0$.

(2) The sphere of radius r in spherical coordinates:

$$\varphi(u,v) = r(\cos u \sin v, \sin u \sin v, \cos v) = rU,$$

$$\varphi_u = r(-\sin u \sin v, \cos u \sin v, 0), \quad \varphi_v = r(\cos u \cos v, \sin u \cos v, -\sin v),$$

$$\varphi_{uu} = r(-\cos u \sin v, -\sin u \sin v, 0), \quad \varphi_{uv} = r(-\sin u \cos v, \cos u \cos v, 0), \quad \varphi_{vv} = -rU = -\varphi.$$

$$E = r^2 \sin^2 v, \quad F = 0, \quad G = r^2, \quad EG - F^2 = r^4 \sin^2 v, \quad l = -r \sin^2 v, \quad m = 0, \quad n = -r,$$

$$K = \frac{1}{r^2}, \quad H = -\frac{1}{r}, \quad k_1 = k_2 = -\frac{1}{r}.$$

(2) The paraboloid $\varphi = (u, v, u^2 + v^2)$:

$$\varphi_u = (1, 0, 2u), \quad \varphi_v = (0, 1, 2v), \quad U = \frac{1}{\sqrt{1 + 4u^2 + 4v^2}}(-2u, -2v, 1),$$

$$\varphi_{uu} = \varphi_{vv} = (0, 0, 2), \quad \varphi_{uv} = (0, 0, 0), \quad E = 1 + 4u^2, \quad F = 4uv, \quad G = 1 + 4v^2,$$

$$l = n = \frac{2}{\sqrt{1 + 4u^2 + 4v^2}}, \quad m = 0, \quad K = \frac{4}{(1 + 4u^2 + 4v^2)^2}, \quad H = \frac{1}{(1 + 4u^2 + 4v^2)^{\frac{1}{2}}} + \frac{1}{(1 + 4u^2 + 4v^2)^{\frac{3}{2}}}.$$

(4) Read other examples in the book.

(5) Surfaces of revolution: Recall that these are obtained by rotating a plane curve around an axis. The circles obtained from rotating a fixed point on the curve are called *parallels* and the curves that are copies (i.e., images by a rotation around the axis of the surface) of the original plane curve are called *meridians*. Suppose that we rotate the curve $\alpha(u) = (g(u), h(u))$ in the x, y-plane around the x-axis. Then a patch for this surface is $\varphi(u,v) = (g(u), h(u)\cos v, h(u) \sin v)$. We have

$$\varphi_u = (g', h' \cos v, h' \sin v), \quad \varphi_v = (0, -h \sin v, h \cos v), \quad \varphi_u \times \varphi_v = (hh', -g'h \cos v, -g'h \sin v)$$

$$U = \frac{\pm 1}{\sqrt{g'^2 + h'^2}}(h', -g' \cos v, -g' \sin v),$$

$$\varphi_{uu} = (g'', h'' \cos v, h'' \sin v), \quad \varphi_{uv} = (0, -h' \sin v, h' \cos v), \quad \varphi_{vv} = (0, -h \cos v, -h \sin v),$$

$$E = g'^2 + h'^2, \quad F = 0, \quad G = h^2, \quad l = \frac{g''h' - g'h''}{\sqrt{g'^2 + h'^2}}, \quad m = 0, \quad n = \frac{hg'}{\sqrt{g'^2 + h'^2}}.$$

So

$$K = \frac{g'(g''h' - g'h'')}{h(g'^2 + h'^2)^2}, \quad H = \frac{h(g''h' - g'h'') + g'(g'^2 + h'^2)}{2h(g'^2 + h'^2)^{3/2}}.$$

If we have arclength parametrization, then

$$E = 1, \quad F = 0, \quad G = h^2, \quad l = g''h' - g'h'', \quad m = 0, \quad n = hg',$$

$$K = \frac{g'(g''h' - g'h'')}{h}, \quad H = \frac{h(g''h' - g'h'') + g'}{2h}.$$

$$g'g''h' - g'^2 h'' = g'g''h' - (1 - h'^2)h'' = -h'' + h'(h'h'' + g'g'') = -h''$$

because $0 = (g'^2 + h'^2)' = 2g'g'' + 2h'h''$. So $K = \frac{-h''}{h}$. Also write $H = \frac{1}{2}(g''h' - g'h'' + g'/h)$ and note that
$$\frac{g'}{h}\left(g''h' - h''g'\right) = -\frac{1}{h}\left(g'g''h' - h''g'^2\right) = K.$$
So that $g''h' - g'h''$ and g'/h are the principal curvatures. We could also have computed these using the remark above.

At any point, the meridian is also a normal section, so its curvature is equal to the normal curvature which in this case is one of the principal curvatures up to sign. In the old exercise 1.4.6, they computed the curvature of a plane curve as $|x'y'' - x'y''|$ which agrees with our computation here.

The parallels are circles of radius $h(u)$ and curvature $\frac{1}{h}$. These are principal curves but in general they are not normal curves which is why their curvature does not agree with the normal curvature necessarily. Recall that for a unit speed curve the normal curvature in the direction of its tangent vector is
$$k(\alpha') = S_p(\alpha') \cdot \alpha' = \alpha'' \cdot U = \kappa N \cdot U = \kappa \cos\theta$$
where θ is the angle between the surface normal and the curve normal. Here $\cos\theta = U \cdot N = U \cdot (0, \cos v, \sin v) = \pm g'$. So the normal curvature which is the principal curvature here is $\pm g' \frac{1}{h}$.

If the curve α is a graph, then we have $\alpha(u,v) = (u, h(u))$. In this case
$$E = 1 + h'^2, \quad F = 0, \quad G = h^2, \quad l = \frac{-h''}{\sqrt{1+h'^2}}, \quad m = 0, \quad n = \frac{-h''}{\sqrt{1+h'^2}}$$
$$K = -\frac{h''}{h(1+h'^2)^2}, \quad H = \frac{1}{2}\left(\frac{-h''}{(1+h'^2)^{3/2}} + \frac{1}{h(1+h'^2)^{1/2}}\right).$$
The first term of H is the curvature of the profile curve or meridian, the second curvature is the normal curvature of the circle or parallel.

12. Dependence of Gaussian curvature on the metric alone

Theorem 12.1. *(Gauss' Theorema Egregium = Remarkable Theorem)*
Suppose given two surfaces M, M' with points $p \in M, q \in M'$, neighborhoods U of p and V of q and a smooth map $\psi : U \to V$ such that $\psi(p) = q$ and ψ is a local isometry at p. Then the Gaussian curvature of M at p is equal to the Gaussian curvature of M' at q.

In other words, the Gaussian curvature of locally isometric surfaces is the same. This is not true for the mean curvature or the principal curvatures. For example, a plane and a cylinder are locally isometric but they do not have the same mean curvature. Two surfaces are locally isometric if we can go from one to the other by bending without stretching.

Theorem 12.2. *(3.4.1) The Gaussian curvature depends on the metric alone. If $F = 0$, then*

$$K = -\frac{1}{2\sqrt{FG}}\left(\frac{\partial}{\partial v}\left(\frac{E_v}{\sqrt{EG}}\right) + \frac{\partial}{\partial u}\left(\frac{G_u}{\sqrt{EG}}\right)\right).$$

To prove the theorem, we need to compute l, m, n, i.e., $\varphi_{uu} \cdot U, \varphi_{uv} \cdot U, \varphi_{vv} \cdot U$ in terms of E, F, G. We introduce the *Christoffel Symbols*. Note that in a patch $\varphi(u,v)$, the vectors φ_u, φ_v, U form a basis. So we can write

$$\begin{aligned} \varphi_{uu} &= \Gamma^u_{uu}\varphi_u + \Gamma^v_{uu}\varphi_v + lU, \\ \varphi_{uv} &= \Gamma^u_{uv}\varphi_u + \Gamma^v_{uv}\varphi_v + mU, \\ \varphi_{vv} &= \Gamma^u_{vv}\varphi_u + \Gamma^v_{vv}\varphi_v + nU. \end{aligned} \tag{12.1}$$

Proof in the book:

Since $F = 0$, we have $\varphi_{uu} \cdot \varphi_u = \Gamma^u_{uu} E$, so $\Gamma^u_{uu} = \frac{\varphi_{uu}\cdot\varphi_u}{E}$. Since $E = \varphi_u \cdot \varphi_u$, we have $E_u = 2\varphi_{uu} \cdot \varphi_u$ and

$$\Gamma^u_{uu} = \frac{E_u}{2E}.$$

Since $0 = F = \varphi_u \cdot \varphi_v$, we have $0 = F_u = \varphi_{uu} \cdot \varphi_v + \varphi_u \cdot \varphi_{uv}$ or $\varphi_{uu} \cdot \varphi_v = -\varphi_u \cdot \varphi_{uv}$.
Furthermore, $E_v = 2\varphi_u \cdot \varphi_{uv} = -2\varphi_{uu} \cdot \varphi_v$. Therefore

$$\Gamma^v_{uu} = \frac{\varphi_{uu} \cdot \varphi_v}{G} = \frac{-E_v}{2G}, \quad \Gamma^u_{uv} = \frac{\varphi_{uv} \cdot \varphi_u}{E} = \frac{E_v}{2E}.$$

Continuing, $G = \varphi_v \cdot \varphi_v$ and $G_u = 2\varphi_v \cdot \varphi_{uv}$. Also, $0 = F = F_v$ gives $-\varphi_v \cdot \varphi_{uv} = \varphi_u \cdot \varphi_{vv}$. So

$$\Gamma^v_{uv} = \frac{\varphi_v \cdot \varphi_{uv}}{G} = \frac{G_u}{2G}, \quad \Gamma^u_{vv} = \frac{\varphi_u \cdot \varphi_{vv}}{E} = \frac{G_u}{2E}.$$

Finally, $G_v = 2\varphi_v \cdot \varphi_{vv}$ and

$$\Gamma^v_{vv} = \frac{\varphi_v \cdot \varphi_{vv}}{G} = \frac{G_v}{2G}.$$

The fundamental acceleration formulas (when $F = 0$):

$$\begin{aligned} \varphi_{uu} &= \tfrac{E_u}{2E}\varphi_u - \tfrac{E_v}{2G}\varphi_v + lU \\ \varphi_{uv} &= \tfrac{E_v}{2E}\varphi_u + \tfrac{G_u}{2G}\varphi_v + mU \\ \varphi_{uu} &= -\tfrac{G_u}{2E}\varphi_u + \tfrac{G_v}{2G}\varphi_v + nU \\ U_u &= -\tfrac{l}{E}\varphi_u + \tfrac{m}{G}\varphi_v \\ U_v &= -\tfrac{m}{E}\varphi_u + \tfrac{n}{G}\varphi_v \end{aligned}$$

A slightly different way:

Dot equations (12.1) with φ_u, φ_v:

(12.2)
$$
\begin{aligned}
\varphi_{uu} \cdot \varphi_u &= \Gamma^u_{uu} E + \Gamma^v_{uu} F = \frac{1}{2} E_u, \\
\varphi_{uu} \cdot \varphi_v &= \Gamma^u_{uu} F + \Gamma^v_{uu} G = F_u - \frac{1}{2} E_v, \\
\varphi_{uv} \cdot \varphi_u &= \Gamma^u_{uv} E + \Gamma^v_{uv} F = \frac{1}{2} E_v, \\
\varphi_{uv} \cdot \varphi_v &= \Gamma^u_{uv} F + \Gamma^v_{uv} G = \frac{1}{2} G_v, \\
\varphi_{vv} \cdot \varphi_u &= \Gamma^u_{vv} E + \Gamma^v_{vv} F = F_v - \frac{1}{2} G_u \\
\varphi_{vv} \cdot \varphi_v &= \Gamma^u_{vv} F + \Gamma^v_{vv} G = \frac{1}{2} G_v.
\end{aligned}
$$

We can rewrite the above as matrix identities:

$$
\begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \Gamma^u_{uu} \\ \Gamma^v_{uu} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} E_u \\ F_u - \frac{1}{2} E_v \end{pmatrix}, \begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \Gamma^u_{uv} \\ \Gamma^v_{uv} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} E_v \\ \frac{1}{2} G_v \end{pmatrix}, \begin{pmatrix} E & F \\ F & G \end{pmatrix}\begin{pmatrix} \Gamma^u_{uu} \\ \Gamma^v_{uu} \end{pmatrix} = \begin{pmatrix} F_v - \frac{1}{2} G_u \\ \frac{1}{2} G_v \end{pmatrix},
$$

or

$$
\begin{pmatrix} \Gamma^u_{uu} \\ \Gamma^v_{uu} \end{pmatrix} = \frac{1}{EG - F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} \frac{1}{2} E_u \\ F_u - \frac{1}{2} E_v \end{pmatrix},
$$

$$
\begin{pmatrix} \Gamma^u_{uv} \\ \Gamma^v_{uv} \end{pmatrix} = \frac{1}{EG - F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} \frac{1}{2} E_v \\ \frac{1}{2} G_v \end{pmatrix},
$$

$$
\begin{pmatrix} \Gamma^u_{uu} \\ \Gamma^v_{uu} \end{pmatrix} = \frac{1}{EG - F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} F_v - \frac{1}{2} G_u \\ \frac{1}{2} G_v \end{pmatrix}.
$$

Substituting in the identities $(\varphi_{uu})_v = (\varphi_{uv})_u, (\varphi_{vv})_u = (\varphi_{uv})_v, U_{uv} = U_{vu}$ in the fundamental acceleration formulas, gives three linear equations of the form $A\varphi_u + B\varphi_v + CU = 0$, hence eight identities $A = 0, B = 0, C = 0$ involving E, F, G, l, m, n and their derivatives. In fact only three (any three) of these are independent, the others give dependency relations, i.e., constraints, between the coefficients E, F, G of the metric.

If we start with smooth functions E, F, G, l, m, n, a surface with these as invariants exists if and only if the above equations hold. This is unlike the case of curves where we saw that for any choice of κ and τ, a curve with curvature κ and torsion τ exists.

13. The importance of the metric or first fundamental form

Given E, F, G we can compute lengths, angles and areas on M without using the coordinates x, y, z on $\mathbb{R}^3$. Recall that φ_u, φ_v form a basis of the tangent space to M at p. For any tangent vector $v = \lambda\varphi_u + \mu\varphi_v$, its length is $|v| = \sqrt{v \cdot v}$ and

$$v \cdot v = (\lambda\varphi_u + \mu\varphi_v)(\lambda\varphi_u + \mu\varphi_v) = \lambda^2 E + 2\lambda\mu F + \mu^2 F.$$

More generally, for two vectors $v = \lambda_1\varphi_u + \mu_1\varphi_v, w = \lambda_2\varphi_u + \mu_2\varphi_v$, their dot product is

$$(\lambda_1\varphi_u + \mu_1\varphi_v)(\lambda_2\varphi_u + \mu_2\varphi_v) = \lambda_1\lambda_2 E + (\lambda_1\mu_2 + \lambda_2\mu_1) F + \mu_1\mu_2 F = \begin{pmatrix} \lambda_1 & \mu_1 \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \lambda_2 \\ \mu_2 \end{pmatrix}.$$

Given a curve $\alpha(t) = \varphi(u(t), v(t))$ on M, by the chain rule,

$$\alpha'(t) = u'(t)\varphi_u + v'(t)\varphi_v.$$

So we can compute its length from $t = a$ to $t = b$:

$$s = \int_a^b |\alpha'(t)| dt = \int_a^b \sqrt{\alpha'(t) \cdot \alpha'(t)} dt = \int_a^b \sqrt{Eu'(t)^2 + 2Fu'(t)v'(t) + Gv'(t)^2} dt.$$

The "line element"

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

is the length of a short curve at p in the direction $\varphi_u du + \varphi_v dv$. If v is constant and we go from (u_0, v_0) to $(u_0 + h, v_0)$, the length is

$$\int_0^1 \sqrt{E(u_0 + ht, v_0) h^2} dt \approx |h| \sqrt{E(u_0, v_0)}.$$

This allows us to estimate E. We can similarly estimate G with a curve where v is constant and F with a curve like $(u_0 + ht, v_0 + ht)$.

We can also use the first fundamental form to compute area on a surface: In the u, v plane a small rectangle of sides du, dv has area $dudv$. It maps via φ to the parallelogram of sides $\varphi_u du, \varphi_v dv$. The area of this parallelogram is (Exercise 1.3.8)

$$|\varphi_u \times \varphi_v| dudv.$$

So, given a region R in the u, v domain, the area of the image of R by φ is

$$\iint_R |\varphi_u \times \varphi_v| dudv.$$

By Lagrange's identity (Exercise 3.1.5), we have

$$0 < |\varphi_u \times \varphi_v|^2 = (\varphi_u \times \varphi_v) \cdot (\varphi_u \times \varphi_v) = (\varphi_u \cdot \varphi_u)(\varphi_v \cdot \varphi_v) - (\varphi_u \cdot \varphi_v)^2 = EG - F^2 = \begin{vmatrix} E & F \\ F & G \end{vmatrix}.$$

So the area is

$$\iint_R \sqrt{EG - F^2}\, dudv.$$

Example: For the sphere we computed $E = r^2 \sin u, F = 0, G = r^2$. So the total area of the sphere is

$$\int_0^{2\pi} dv \int_0^{\pi} r^2 \sin u \, du = 4\pi r^2.$$

14. Geodesics

Given a curve $\alpha(t) = \varphi(u(t), v(t))$ on a surface M, the acceleration splits into its tangential and normal components:

$$\alpha'' = \alpha''_{norm} + \alpha''_{tan} = (\alpha'' \cdot U)U + \alpha''_{tan} = (\alpha'' \cdot U)U + \lambda_t \varphi_u + \mu_t \varphi_v.$$

We usually have the basis φ_u, φ_v for the tangent space to M. We can also make another basis adapted to the curve α. We take as the first basis vector the unit tangent T for the curve α and as second basis vector the cross-product $U \times T$. In this way we have the orthonormal basis $T, U \times T, U$:

$$\alpha'' = (\alpha'' \cdot T)T + (\alpha'' \cdot U \times T)U \times T + (\alpha'' \cdot U)U.$$

So

$$\alpha''_{tan} = (\alpha'' \cdot T)T + (\alpha'' \cdot U \times T)U \times T, \quad \alpha''_{norm} = (\alpha'' \cdot U)U.$$

If α has constant speed, then $\alpha' \cdot \alpha'$ is constant, hence its derivative is zero and $\alpha' \cdot \alpha'' = 0$. So, if α has constant speed, then its component on T is zero and

$$\alpha'' = (\alpha'' \cdot U \times T)U \times T + (\alpha'' \cdot U)U.$$

Recall that if α is a unit speed curve, its curvature is $\kappa = |\alpha''|$ and the normal curvature along α is

$$k_n := S_p(\alpha') \cdot \alpha' = \alpha'' \cdot U.$$

When α has unit speed, the tangential or geodesic curvature of α is

$$k_g := \alpha'' \cdot U \times T = \pm|\alpha''_{tan}|$$

and we have

$$\kappa^2 = k_n^2 + k_g^2.$$

We can also write

$$\alpha'' \cdot U \times T = \alpha'' \cdot U \times \alpha' = U \cdot \alpha' \times \alpha'' = |\alpha' \times \alpha''| \cos\theta = \kappa \cos\theta$$

where θ is the angle between $\alpha' \times \alpha'' = T \times \kappa N = \kappa B$ and U.

Definition 14.1. *A curve is called a geodesic if* $\alpha''_{tan} = 0$.

Note that geodesics have constant speed (5.1.7), since, if $\alpha''_{tan} = 0$, then α'' is parallel to U, hence perpendicular to α' and $(\alpha' \cdot \alpha')' = 2\alpha' \cdot \alpha'' = 0$.

Now write $\alpha(t) = \varphi(u(t), v(t))$,

$$\alpha'(t) = u'\varphi_u + v'\varphi_v,$$

$$\alpha''(t) = u''\varphi_u + u'\left(u'\varphi_{uu} + v'\varphi_{uv}\right) + v''\varphi_v + v'\left(u'\varphi_{uv} + v'\varphi_{vv}\right),$$

$$\alpha''_{tan} = u''\varphi_u + u'^2\left(\Gamma^u_{uu}\varphi_u + \Gamma^v_{uu}\varphi_v\right) + v''\varphi_v + 2u'v'\left(\Gamma^u_{uv}\varphi_u + \Gamma^v_{uv}\varphi_v\right) + v'^2\left(\Gamma^u_{vv}\varphi_u + \Gamma^v_{vv}\varphi_v\right).$$

$$\alpha''_{tan} = \left(u'' + u'^2\Gamma^u_{uu} + 2u'v'\Gamma^u_{uv} + v'^2\Gamma^u_{vv}\right)\varphi_u + \left(u'^2\Gamma^v_{uu} + v'' + 2u'v'\Gamma^v_{uv} + v'^2\Gamma^v_{vv}\right)\varphi_v.$$

So α is a geodesic if and only if

$$u'' + u'^2\Gamma^u_{uu} + 2u'v'\Gamma^u_{uv} + v'^2\Gamma^u_{vv} = u'^2\Gamma^v_{uu} + v'' + 2u'v'\Gamma^v_{uv} + v'^2\Gamma^v_{vv} = 0.$$